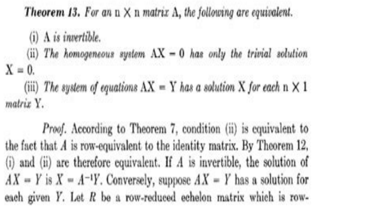



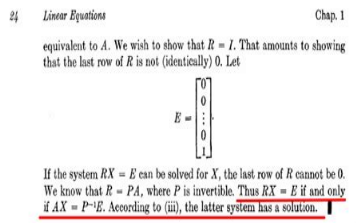
Theorem 13. *For an* n × n *matrix* A, *the following are equivalent.*

(i) A *is invertible.*

(ii) *The homogeneous system* AX = 0 *has only the trivial solution* X = 0.

(iii) *The system of equations* AX = Y *has a solution* X *for each* n × 1 *matrix* Y.

Proof. According to Theorem 7, condition (ii) is equivalent to the fact that A is row-equivalent to the identity matrix. By Theorem 12, (i) and (ii) are therefore equivalent. If A is invertible, the solution of $AX = Y$ is $X = A^{-1}Y$. Conversely, suppose $AX = Y$ has a solution for each given Y. Let R be a row-reduced echelon matrix which is row-



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equivalent to A. We wish to show that $R = I$. That amounts to showing that the last row of R is not (identically) 0. Let

$$E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

If the system $RX = E$ can be solved for X, the last row of R cannot be 0. We know that $R = PA$, where P is invertible. Thus $RX = E$ if and only if $AX = P^{-1}E$. According to (iii), the latter system has a solution. ▌