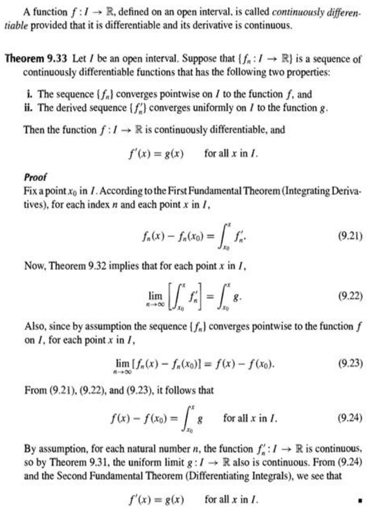

A function $f : I \to \mathbb{R}$, defined on an open interval, is called *continuously differentiable* provided that it is differentiable and its derivative is continuous.

Theorem 9.33 Let I be an open interval. Suppose that $\{f_n : I \to \mathbb{R}\}$ is a sequence of continuously differentiable functions that has the following two properties:

i. The sequence $\{f_n\}$ converges pointwise on I to the function f, and
ii. The derived sequence $\{f'_n\}$ converges uniformly on I to the function g.

Then the function $f : I \to \mathbb{R}$ is continuously differentiable, and

$$f'(x) = g(x) \qquad \text{for all } x \text{ in } I.$$

Proof
Fix a point x_0 in I. According to the First Fundamental Theorem (Integrating Derivatives), for each index n and each point x in I,

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} f'_n. \qquad (9.21)$$

Now, Theorem 9.32 implies that for each point x in I,

$$\lim_{n\to\infty}\left[\int_{x_0}^{x} f'_n\right] = \int_{x_0}^{x} g. \qquad (9.22)$$

Also, since by assumption the sequence $\{f_n\}$ converges pointwise to the function f on I, for each point x in I,

$$\lim_{n\to\infty}[f_n(x) - f_n(x_0)] = f(x) - f(x_0). \qquad (9.23)$$

From (9.21), (9.22), and (9.23), it follows that

$$f(x) - f(x_0) = \int_{x_0}^{x} g \qquad \text{for all } x \text{ in } I. \qquad (9.24)$$

By assumption, for each natural number n, the function $f'_n : I \to \mathbb{R}$ is continuous, so by Theorem 9.31, the uniform limit $g : I \to \mathbb{R}$ also is continuous. From (9.24) and the Second Fundamental Theorem (Differentiating Integrals), we see that

$$f'(x) = g(x) \qquad \text{for all } x \text{ in } I. \quad \blacksquare$$